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Controlled synchronization of chaotic systems with uncertainties via a sliding mode control design

Samuel Bowong*
INRIA Lorraine, Project CONGE and University of Metz, ISGMP, Bâtiment A, Ile de Saulcy, 57045 Metz, Cedex 01, France

F. M. Moukam Kakmeni† and Clement Tchawoua‡
Laboratoire de Mécanique, Département de Physiques, Faculté des Sciences, Université de Yaoundé I, B.P. 812, Yaoundé, Cameroun
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This paper addresses the problem of robust adaptive control for synchronization of continuous-time coupled chaotic systems with uncertainties. A general model is studied using measured output state feedback control. An adaptive controller is designed based on a sliding mode control design. When only the output variable is measurable for synchronization, the adaptive controller is designed by incorporating with an observer. Two numerical examples are presented to show the effectiveness of the proposed chaos synchronization method.

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I. INTRODUCTION

Motivated by the study of chaotic phenomena (see, e.g., [1,2]), recent years have seen an increase in the interest in synchronization. The idea of synchronizing two identical systems that start from different initial conditions was introduced by Pecora and Caroll [3]. It investigates the linking of the trajectory of one system to the other system with the same values parameter (solution), such that they remain together in each step through the transmission of a signal. Adhering to the Pecora-Caroll drive-response concept, several chaos synchronization schemes have been successfully established [3–7]. But to have an exact synchronous system, a response system must have an identical copy of the chaotic system used by the drive system. However, it is practically impossible to have two identical chaotic systems because of impossible-to-avoid tolerances in the real-world physical control parameters. For example, the tolerances of the various electric elements in two "identical" electronic circuits will certaintly lead to small but definite differences in the physical parameters and thus the operations of the circuits. Thus by definition synchronization cannot occur. It has been proposed that one replace it with a generalized (or practical) synchronization.

Recently, specialists from (nonlinear) control theory turned their attention to the study of controlled synchronization. It has been demonstrated for important special cases of "drive/response" and coupled systems that synchronizing control may be designed using feedback linearization or passification methods. Incomplete information about the system parameters has been taken into account (adaptive and robust synchronization [7–11]) as well as incomplete information of the state of the system (observer-based synchronization [12]). Although the parameter structure can be known in some cases, it would be desirable to have a feedback scheme to achieve synchronization in spite of the slave oscillator having the least prior knowledge about the structure of the master system. This necessity for robustness can be required in some systems (for instance, the multimode laser, animal gait, or oscillatory neural systems). Thus, it is important and necessary to design robust synchronization schemes for uncertain chaotic systems with various types of uncertainties.

As an alternative, in recent years sliding mode control has received much attention and become an active research area. Sliding mode control is a nonlinear control strategy. It includes two parts: switched control and equivalent control. Traditional sliding mode control cannot assure system robustness. It also often has chattering phenomena for which the sliding mode controller is applied. Therefore, designing a sliding mode controller involves selecting a correct update law or a suitable switching manifold (e.g., [14]). Sliding mode control has a learning capability for improving the feedback control performance by incorporating useful updated information on line. An advantage of sliding mode control, on the other hand, is the system robustness with respect to certain system parameter variations and external disturbances [13–15]. Since sliding mode control is suitable for the synchronization of uncertain nonlinear systems, it is a good candidate for synchronizing uncertain chaotic systems [16].

Previous work of others [17–19] has presented interesting results on the synchronization of chaotic systems based on the sliding mode control design. In particular, the authors of [19] studied the synchronization of chaotic systems with uncertainties. Although their analysis ensured synchronization stability, it is not clear when synchronization can be achieved for a given estimation of the system's uncertainties and how to choose the estimated value of the uncertainties. This is because the estimated value of the system model uncertainties depends on the state of the observer in a complex manner.

We believe that this difficulty to estimate the system's uncertainties arises because the system structure information

*Author to whom correspondence should be addressed. Permanent address: B.P. 8329, Yaoundé, Cameroun. FAX: (237) 231-95-84. Email address: sbowong@uycdc.uninet.cm
†Email address: fmoukam@uycdc.uninet.cm
‡Email address: ctchawoua@uycdc.uninet.cm

$\Xi(\cdot)$ involving the modeling differences, uncertain parameters, and the unknown external perturbations has been used in the sliding control law [see Eq. (14) in Yang and Shao's paper]. In principle, an analysis that focuses on geometrical control theory [7] could exploit the specific property and structure of the system to provide a dynamic estimation of the system's uncertainties.

This paper, in the spirit of previous works [17–19], studies the synchronization of chaotic systems with uncertainties based on a sliding mode control design. The proposed scheme consists of a dynamical output feedback which performs the suppression of chaos on the uncertain system. The main idea behind our proposal is, departing from the uncertain system, to construct an extended nonlinear system which should be dynamically equivalent to the canonical representation. In this way, the system's uncertainties are lumped into a nonlinear function, which is rewritten into the extended nonlinear system as a state variable. By using the results reported by Teel and Praly [20], an observer can be constructed to get an estimated value of the lumping nonlinear function via the augmented state variable. More importantly, we show how the convergence rate can be assigned by tuning the switching parameter of the sliding surface. Numerical simulations on the synchronization of ϕ^6-Duffing and ϕ^6–Van der Pol oscillators as well as two Lorenz systems with parameter mismatching are used to illustrate our findings.

With respect to existing results in the literature (see [17–19]), our contribution in this paper can be summarized as follows.

(i) An explicit construction of the estimation of the system's uncertainties is provided, while in [19] the procedure to estimate the system model uncertainties is not clear.

(ii) A prescribed synchronization error convergence rate can be assigned. This can be done easily by tuning a single parameter of the sliding surface. To the best of author's knowledge, this issue has not been previously studied.

The paper is organized as follows. In Sec. II, the class of uncertain chaotic systems is established. In Sec. III, the problem of chaotic system synchronization based on a sliding mode control design is analyzed. In Sec. IV, some numerical simulations to illustrate our findings are carried out, and finally in Sec. V, some conclusions are presented.

II. A CLASS OF UNCERTAIN CHAOTIC SYSTEMS

To investigate synchronization, we consider two nonidentical chaotic systems with uncertainties. The drive system is

$$\dot{x} = f(x,t),$$

$$y_d = C_d x, \tag{1}$$

where $x \in \mathbb{R}^n$ is a vector of drive system states, f is a nonlinear vector function defining the flow, and $y_d \in \mathbb{R}$ denotes the measurable output state that can be transmitted. C_d is a vector of proper length which defines the output channel. Note that, without loss of generality, we can assume that the measured state is given by $y_d = x_1$. For the following, it is assumed that system (1) exhibits chaotic dynamics. The response system is

$$\dot{y} = g(y,t) + Bu,$$

$$y_r = C_r y, \tag{2}$$

where $y \in \mathbb{R}^n$ is a vector of the response system states, g is a nonlinear vector function defining the flow, B is a constant matrix which defines the control channel, and $u \in \mathbb{R}$ is a control input which has to be chosen. The vector C_r defines the measured state of the response system. Note that we can assume that $C_d = C_r$. The role of the feedback u is thus to force the convergence of the response towards the drive orbit. To carry out such an investigation, let us introduce the variable $e = y - x$, which is the measure of the nearness of the response to the drive. Introducing e in Eq. (2), we obtain the following equation:

$$\dot{e} = g(e + x,t) - f(x,t) + Bu,$$

$$y_e = Ce. \tag{3}$$

The synchronization is achieved when e goes to zero as t increases or, practically, is less than a given precision.

Now, let us define a coordinates transformation $z = \Phi(e)$ such that the error system (3) can be globally transformed into the canonical form [21],

$$\dot{z}_i = z_{i+1},\ i = 1,2,\ \ldots,\rho - 1,$$

$$\dot{z}_\rho = \alpha(z,\nu) + \gamma(z,\nu)u,$$

$$\dot{\nu} = \zeta(z,\nu),$$

$$\tilde{y} = z_1, \tag{4}$$

where $\tilde{y}$ is the system output, ρ is the relative degree of the error system (i.e., the lowest-order time derivative such that the control command u is directly related to the output $\tilde{y}$), and $\nu \in \mathbb{R}^{n-\rho}$ is the unobservable states vector (internal dynamics).

Often, the constant matrix B can be selected to make possible the transform from Eq. (3) to the assumed format (4). System (4), which is very general, contains most well known chaotic systems and some special models for chaotic systems like those proposed in [13]. For example, the Lorenz dynamical, the Rössler system, and several types of Chua's circuits can be transformed into the canonical form with a relative degree $\rho < n$. On the other hand, a nonautonomous second-order chaotic system such as the Duffing oscillator and the Van der Pol system can be written as the canonical form with $\rho = n$. In addition, if $\rho = n$, the transformed system (4) is the so-called fully linearizable nonlinear system, and if $\rho < n$, the system (4) is called a partially linearizable nonlinear system.

Nevertheless, if the vector fields $f(x,t)$ and $g(y,t)$ are uncertain, the coordinates transformation $z = \Phi(e)$, bringing the error system (3) into the canonical form (4), is uncertain. In principle, since the coordinates transformation is a diffeomorphism, one can suppose that the uncertain transformation exists and it is invertible. However, since $\Phi(e)$ is uncertain, the nonlinear functions $\alpha(z,\nu)$ and $\gamma(z,\nu)$ are also uncertain, hence they cannot be directly used in a linearizing-like feedback.

In the next section, the detailed design procedure of the feedback control law u is described with detailed explanations.

III. SYNCHRONIZATION VIA MEASURED OUTPUT FEEDBACK

In this section, we present a robust control design for uncertain chaotic systems in canonical form (4). The main idea is to construct a dynamically equivalent system which is itself uncertain. Dynamic output feedback is applied to perform chaos synchronization in spite of modeling differences, parameter variations, and noisy measurements.

To describe the new design and analysis, the following assumptions are needed: (A_1) Only the system output $\tilde{y}=z_1$ is available for feedback. (A_2) $\gamma(z)$ is bounded away from zero. However, an estimate $\hat{\gamma}(z)$ of $\gamma(z,\nu)$, satisfying $\mathrm{sgn}[\hat{\gamma}(z)]=\mathrm{sgn}[\gamma(z,\nu)]$, is available for feedback. (A_3) System (5) is the minimum phase, i.e., the subsystem of the zero dynamics, $\dot{\nu}=\zeta(0,\nu)$, where $\nu\in\mathbb{R}^{n-\rho}$, is asymptotically stable. The first assumption is realistic because in many applications, due to the difficulty in measurements or demand for security, some state variables cannot be measured. For example, in secure communication, a signal is transmitted from the drive system to the response system, where the transmitted signal, which is kept unmeasurable, is part of state variables. The second assumption implies that the origin is not a singularity point when a linearizing-like feedback is used to perform synchronization. The minimum phase supposition is a stronger condition, which implies that the uncontrollable states $\nu\in\mathbb{R}^{n-\rho}$ of the uncertain system are asymptotically stable. This is reasonable for the boundness of the chaotic attractor in state space and the interaction of all trajectories inside the attractor. So when we taken actions to achieve $\lim_{t\to\infty} z_i=0$, $i=1,\ldots,\rho$, the part $\zeta(0,\nu)\to\zeta(0,\nu)\to 0$ asymptotically for the so-called minimum phase character. Fortunately, most chaotic oscillators satisfy this assumption.

In order to determine the sliding mode control law, the reformulation of the state space equation of system (4) into an extended controllable canonical form is required. To this end, let us define

$$\delta(z)=\gamma(z,\nu)-\hat{\gamma}(z),\ \ \Theta(z,\nu,u)=\alpha(z,\nu)+\delta(z,\nu)u,$$

$$z_{n+1}=\Theta(z,\nu,u)+\hat{\gamma}(z)u,$$

and

$$\eta=\sum_{k=1}^{n} z_{k+1}\partial_k z_{n+1}+\delta(z,\nu)\dot{u}+\zeta(z,\nu)\partial_\nu z_{n+1} \qquad (5)$$

with $\partial_k z_{n+1}=\partial z_{n+1}/\partial z_k$, $k=1,2,\ldots,n$, and $\partial_\nu z_{n+1}=\partial z_{n+1}/\partial\nu$.

Then, there exists a time-invariant manifold $\psi(z,z_{n+1},\eta,u,\dot{u},\ddot{u})=0$ such that the solution of system (4) is a projection of the solution of the following dynamical system:

$$\begin{aligned}
&\dot{z}_i=z_{i+1},\ 1\leqslant i\leqslant n,\\
&\dot{z}_{n+1}=\eta+\hat{\gamma}(z)\dot{u},\\
&\dot{\eta}=\Xi(z,z_{n+1},\eta,\nu,u,\dot{u},\ddot{u}),\\
&\dot{\nu}=\zeta(z,\nu),\\
&\tilde{y}=z_1,
\end{aligned} \qquad (6)$$

where

$$\begin{aligned}
\Xi(z,z_{n+1},\eta,\nu,u,\dot{u},\ddot{u}) &= \sum_{k=1}^{n} z_{k+1}^2\partial_k z_{n+1}+\sum_{k=1}^{n-1} z_{k+2}\partial_k z_{n+1}\\
&+[\eta+\hat{\gamma}(z)]\partial_n z_{n+1}\\
&+\sum_{k=1}^{n} z_{k+1}\partial_k[\zeta(z,\nu)\partial_\nu\Theta(z,\nu,u)]\\
&+\zeta(z,\nu)\partial_\nu[\zeta(z,\nu)\partial_\nu\Theta(z,\nu,u)]\\
&+\dot{u}\Bigg[\delta(z,\nu)+\hat{\gamma}(z)+\zeta(z,\nu)\partial_\nu\delta(z,\nu)\\
&+\sum_{k=1}^{n}\partial_k\delta(z,\nu)\Bigg]+\delta(z,\nu)\ddot{u}
\end{aligned}$$

with $\partial_\nu\Theta(z,\nu,u)=\partial\Theta(z,\nu,u)/\partial\nu$, $\partial_k\delta(z,\nu)=\partial\delta(z,\nu)/\partial z_k$, and $\partial_k\zeta(z,\nu)=\partial\zeta(z,\nu)/\partial z_k$, $k=1,2,\ldots,n$, i.e., system (6) is dynamically equivalent to system (4). It must be pointed out that the manifold $\psi(z,z_{n+1},\eta,\nu,u,\dot{u},\ddot{u})=\eta-\sum_{k=1}^{n} z_{k+1}\partial_k z_{n+1}+\delta(z,\nu)\dot{u}+\zeta(z,\nu)\partial_\nu z_{n+1}=0$ is, by definition, time-invariant. In fact, it is straightforward to prove that the set

$$\begin{aligned}
\Psi&=\{\psi(z,z_{n+1},\eta,\nu,u,\dot{u},\ddot{u})\}\\
&=\eta-\sum_{k=1}^{n} z_{k+1}\partial_k z_{n+1}+\delta(z,\nu)\dot{u}+\zeta(z,\nu)\partial_\nu z_{n+1},
\end{aligned}$$

$$\begin{aligned}
\dot{\eta}&=\sum_{k=1}^{n} z_{k+1}^2\partial_k z_{n+1}+\sum_{k=1}^{n-1} z_{k+2}\partial_k z_{n+1}+[\eta+\hat{\gamma}(z)]\partial_n z_{n+1}\\
&+\sum_{k=1}^{n} z_{k+1}\partial_k[\zeta(z,\nu)\partial_\nu\Theta(z,\nu,u)]\\
&+\zeta(z,\nu)\partial_\nu[\zeta(z,\nu)\partial_\nu\Theta(z,\nu,u)]\\
&+\dot{u}\left[\delta(z,\nu)+\hat{\gamma}(z)+\zeta(z,\nu)\partial_\nu\delta(z,\nu)+\sum_{k=1}^{n}\partial_k\delta(z,\nu)\right]\\
&+\delta(z,\nu)\ddot{u}
\end{aligned}$$

satisfies $d\psi/dt=0$ for all $t\geqslant 0$. Now, from the equality $\psi(z,z_{n+1},\eta,\nu,u,\dot{u},\ddot{u})=0$ and condition $d\psi/dt=0$, one can take the first integral of system (6) to get $\eta=\sum_{k=1}^{n} z_{k+1}\partial_k z_{n+1}+\delta(z,\nu)\dot{u}+\zeta(z,\nu)\partial_\nu z_{n+1}$. When the first integral is backsubstituted into system (6), we obtain the solution of system (4). Hence, the solution of system (4) is a projection of system

(6) via the module $\pi(z,z_{n+1},\eta,\nu)=(z,\nu)$. This is, system (6) is dynamically equivalent to system (4) if initial conditions, $(z(0),z_{n+1}(0),\eta(0),\nu(0))$ are contained in $\psi(z,z_{n+1},\eta,u,\nu,\dot{u},\ddot{u})$, i.e., the augmented state η provides the dynamics of the uncertain function $\Theta(z,\nu,u)$ which involves modeling differences, uncertain parameters, and unknown external disturbances.

The following must be noted. Since the nonlinear functions $\alpha(z,\nu)$ and $\gamma(z,\nu)$ are uncertain, the nonlinear function $\Theta(z,\nu,u)$ is also uncertain, hence it cannot be directly used in a sliding mode control law.

A key feature of Eq. (6) is that the uncertainties have been lumped in an uncertain function $\Theta(z,\nu,u)$ which can be estimated by means of the nonmeasured but observable state η. By an observable state, we mean that the dynamics of such a state can be reconstructed from one-line measurements (for example, $\tilde{y}=z_1$). Furthermore, if one is able to stabilize the system (6) without making use of the constraint $\psi(z(0),z_{n+1}(0),\eta(0),u(0),\nu(0),\dot{u}(0),\ddot{u}(0))=0$, one would be able to stabilize system (4) and its equivalent system (3).

By the concept of extended systems, a suitable sliding surface can be chosen as

$$S = z_{n+1} - z_{n+1}(0) + \int_0^t \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j = 0, \tag{7}$$

where $z_{n+1}(0)$ is the initial state of $z_{n+1}(t)$, $\theta > 0$ is the switching gain which is determined such that the sliding condition is satisfied and sliding mode motion will occur, and K_j, $j=1,2,\ldots,n+1$ are constant parameters of the sliding surface which are computed from the following procedure.

Equation (7) can also be reformulated as

$$\dot{z}_{n+1} = -\sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j. \tag{8}$$

Therefore, the sliding mode dynamics (the desired dynamics) can be described as

$$\begin{aligned} &\dot{z}_i = z_{i+1},\ 1 \leqslant i \leqslant n,\\ &\dot{z}_{n+1} = -\sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j,\\ &\dot{\eta} = \Xi(z,z_{n+1},\eta,\nu,u,\dot{u},\ddot{u}),\\ &\dot{\nu} = \zeta(z,\nu), \end{aligned} \tag{9}$$

or in a matrix equation form as

$$\begin{aligned} &\dot{Z} = \theta \Delta_\theta^{-1} A \Delta_\theta Z,\\ &\dot{\eta} = \Xi(Z,\eta,\nu,u,\dot{u},\ddot{u}),\\ &\dot{\nu} = \zeta(z,\nu), \end{aligned} \tag{10}$$

where $Z=(z,z_{n+1})^T$, $\Delta_\theta=\mathrm{diag}(\theta^{-1},\ldots,\theta^{n})$ with Δ_θ^{-1} its inverse and

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 1\\ -K_1 & -K_2 & -K_3 & \ldots & -K_{n+1} \end{bmatrix}.$$

Then K_j, $j=1,2,\ldots,n+1$ are chosen such that the matrix A has all its eigenvalues at the open left-half complex plane (i.e., all roots of polynomial $s^{n+1}+K_{n+1}s^n+\cdots+K_2 s+K_1=0$ have negative real parts).

The sliding surface used in this paper is one dimension higher than the traditional sliding surface, which guarantees that it passes through the initial states of the system being controlled. The reaching law is chosen as

$$\dot{S} = \beta S - \theta\,\mathrm{sgn}(S), \tag{11}$$

where $0 \leqslant \beta < 1$ and $\mathrm{sgn}(\cdot)$ denotes the signum function. From Eqs. (7) and (11), it can be found that

$$\dot{S} = \beta S - \theta\,\mathrm{sgn}(S) = \dot{z}_{n+1} + \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j, \tag{12}$$

or, alternatively,

$$\dot{z}_{n+1} = \eta + \hat{\gamma}(z)\dot{u} = \beta S - \theta\,\mathrm{sgn}(S) - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j. \tag{13}$$

So if the initial condition $u(0)=0$, then the differential equation of control input u can be determined as

$$\dot{u} = \frac{1}{\hat{\gamma}(z)}\left[\beta S - \theta\,\mathrm{sgn}(S) - \eta - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j\right]. \tag{14}$$

Therefore, the control input can be obtained as

$$u = \int_0^t \left[\frac{1}{\hat{\gamma}(z)}\left(\beta S - \theta\,\mathrm{sgn}(S) - \eta - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j\right)\right] dt, \tag{15}$$

with $u(0)=0$. It should be pointed out that a large θ is important for the realization of synchronization, which is associated with the system information of the two chaotic systems. This question can be qualitatively analyzed with the Lyapunov stability theory as follows.

Substituting the control law of Eq. (14) into the extended system (6), the dynamics of the closed-loop system can be described as

$$\begin{aligned} &\dot{z}_i = z_{i+1},\ i = 1,2,\ldots,n,\\ &\dot{z}_{n+1} = \beta S - \theta\,\mathrm{sgn}(S) - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j,\\ &\dot{\eta} = \Xi(z,z_{n+1},\eta,u,\dot{u},\ddot{u}),\\ &\dot{\nu} = \zeta(z,\nu). \end{aligned} \tag{16}$$

Let the Lyapunov function of the system be

$$V = \frac{1}{2}S^2, \tag{17}$$

therefore V is a positive semidefinite function. The first derivative of V with respect to time is obtained as

$$\begin{aligned}\dot{V} = S\dot{S} &= S\left(\dot{z}_{n+1} - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j z_j\right)\\ &= S[\beta S - \theta\,\mathrm{sgn}(S)]\\ &= \beta S^2 - \theta \mathrm{abs}(S)\\ &\leqslant \mathrm{abs}(S)[\mathrm{abs}(S) - \theta].\end{aligned} \tag{18}$$

If $\dot{V} \leqslant 0$ is satisfied, the sliding mode will exist. From Eq. (7), we know that the sliding surface S depends on (z, z_{n+1}), hence on e or x and y. For the boundness of the chaotic attractor, we know that S is bounded. So a large enough θ will lead to $\dot{V} \leqslant 0$. Convergence of $\eta(t)$ to zero follows from the fact that the closed-loop system is in cascade form. In many situations, the condition $\dot{V} \leqslant 0$ can be satisfied by choosing a large enough switching gain θ. On the other hand, the θ parametrization of the feedback control law (15) provides a simple tuning procedure. In fact, in a matrix equation form, the first equation of the closed-loop system (16) can be rewritten as

$$\dot{Z} = \theta \Delta_\theta^{-1} A \Delta_\theta Z + \Omega(S), \tag{19}$$

where A is defined as in Eq. (10) and $\Omega(S) = [0, \ldots, 0, \beta S - \theta\,\mathrm{sgn}(S)]^T$. The integration of the closed-loop system (19) yields

$$Z(t) = \exp(\theta \Delta_\theta^{-1} A \Delta_\theta t) Z(0) + \exp(\theta \Delta_\theta^{-1} A \Delta_\theta t) \times \int_0^t \exp(-\theta \Delta_\theta^{-1} A \Delta_\theta \sigma)\Omega(S) d\sigma.$$

Since the matrix A is Hurwitz and the surface S is bounded for all $t \geqslant 0$, $\Omega(S)$ is also a bounded function, i.e., $\|\Omega(S)\| \leqslant L$. Then using the triangle and Schwartz inequalities, one has the following inequality:

$$\|Z(t)\| \leqslant \|\exp(\theta \Delta_\theta^{-1} A \Delta_\theta t) Z(0)\| + L \int_0^t \|\exp(-\theta \Delta_\theta^{-1} A \Delta_\theta \sigma) d\sigma\|.$$

Then, the trajectories $Z(t)$ are bounded for all $t \geqslant 0$, that is, $Z(t) \to \mathcal{B}(R(\theta^{-1}))$, where $\mathcal{B}(R(\theta^{-1}))$ is a ball with radius on the order θ^{-1}. In fact, as the switching parameter θ increases, $\|Z(t)\|$ decreases, and the faster the convergence of $Z(t)$ is.

Note that the sliding surface (7) and the sliding controller (15) require full information about the states of system (6). In this sense, the following comments are in order. (i) The augmented states z_{n+1} and η are not available for feedback. This fact is obvious because z_{n+1} and η represent, by definition, the mismatches between the drive and response systems. (ii) It is desired that only one state is available for feedback from one-line measurements. Consequently, estimated values of the states (z, z_{n+1}, η) are required for practical implementation. To this end, the following uncertainty estimator is proposed:

$$\begin{aligned}\dot{\hat{z}}_i &= \hat{z}_{i+1} - \theta^i C_i(\hat{z}_1 - z_1), \ 1 \leqslant i \leqslant n,\\ \dot{\hat{z}}_{n+1} &= \hat{\eta} + \hat{\gamma}(z)\dot{u} - \theta^{n+1} C_{n+1}(\hat{z}_1 - z_1),\\ \dot{\hat{\eta}} &= -\theta^{n+2} C_{n+2}(\hat{z}_1 - z_1),\end{aligned} \tag{20}$$

where C_j, $j = 1, 2, \ldots, n+2$ are estimation parameters. Appropriately choosing parameters C_j, $j = 1, 2, \ldots, n+2$, $(\hat{z}_1, \ldots, \hat{z}_n)$, $\hat{z}_{n+1}$, and $\hat{\eta}$ will converge to z_j, $1 \leqslant j \leqslant n$, z_{n+1}, and η, respectively. Note that since $\Theta(z, \nu, u)$ is uncertain, the function $\Xi(z, z_{n+1}, \eta, \nu, u, \dot{u}, \ddot{u})$ correspondingly is unknown. Thus, such a term has been neglected in the construction of the observer (20).

In order to determine C_j, $j = 1, 2, \ldots, n+1$, let $\tilde{e} \in \mathbb{R}^{n+2}$ be an estimation error vector whose components are defined as follows: $\tilde{e}_i = \theta^{n+1-i}(\hat{z}_i - z_i)$, $i = 1, 2, \ldots, n$ and $\tilde{e}_{n+2} = \hat{\eta} - \eta$. Then, the dynamics of the estimation error can be written as follows:

$$\dot{\tilde{e}} = \theta D \tilde{e} + \Gamma(z, z_{n+1}, \nu, \eta, u, \dot{u}, \ddot{u}), \tag{21}$$

where

$$\Gamma(z, z_{n+1}, \eta, u, \nu, \dot{u}, \ddot{u}) = [0, 0, \ldots, 0, \Xi(z, z_{n+1}, \eta, \nu, u, \dot{u}, \ddot{u})]^T$$

and the matrix D is given by

$$D = \begin{bmatrix} -C_1 & 1 & 0 & \ldots & 0\\ -C_2 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -C_{n+1} & 0 & 0 & \ldots & 1\\ -C_{n+2} & 0 & 0 & \ldots & 0\end{bmatrix}.$$

The estimation parameters C_j, $1 \leqslant j \leqslant n+2$ are chosen in such a way that the polynomial $P_{n+2}(s) = s^{n+2} + C_1 s^{n+1} + \cdots + C_{n+1}s + C_{n+2}$ is Hurwitz. In addition, since the trajectories $x(t)$ and $y(t)$ are contained in some chaotic attractor, then $\Xi(z, z_{n+1}, \eta, u, \nu, \dot{u}, \ddot{u})$ is a bounded function. Consequently, after choosing C_j, $j = 1, 2, \ldots, n+2$ so that all the eigenvalues of D are located in the left-half complex plane, we can conclude that for a sufficiently large value of $\theta > 0$, $\tilde{e} \to 0$ as $t \to \infty$, which implies that $\hat{z}_j \to z_j$, $j = 1, 2, \ldots, n$, $\hat{z}_{n+1} \to z_{n+1}$, and $\hat{\eta} \to \eta$. So we can get the information of unmeasurable states from $\hat{z}_i$, $i = 1, 2, \ldots, n$ and the model uncertainties from $\hat{z}_{n+1}$ and $\hat{\eta}$. Then, the sliding surface (7) and the sliding control law (15) become

$$S = \hat{z}_{n+1} - \hat{z}_{n+1}(0) + \int_0^t \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j \hat{z}_j = 0 \tag{22}$$

and

$$u = \int_0^t \left(\frac{1}{\hat{\gamma}(\hat{z})} \left[\beta S - \theta\,\mathrm{sgn}(S) - \hat{\eta} - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j \hat{z}_j \right) \right] dt \tag{23}$$

with $u(0) = 0$.

Notice that the sliding surface (22) and the sliding controller (23) only use estimated values of the uncertain terms $\alpha(z,\nu)$ and $\gamma(z,\nu)$ (by means z_{n+1} and $\hat{\eta}$) and $\hat{z}$. So Eqs. (22) and (23) neglect the system uncertainties and are more physically realizable than Eqs. (7) and (15). Thus, the robust exponential stabilization is given by the dynamic compensator (20), the sliding surface (22), and the sliding control law (23).

The proposed controller has the following advantages regarding the adaptive control schemes: (i) the order of the proposed controller does not increase with the number of parameters; (ii) if the system is nonlinear in its parameter structure, the proposed controller does not change because the controller does not require information about system parameters; and (iii) a large θ in the sliding surface will increase the robustness of adaptive control, while a small θ will be good for robust stability. Therefore, in practice, a trade-off will be made according to the purpose of the design. As a result, the robust feedback controller (20), (22), and (23) can be experimentally implemented to perform chaos synchronization on a class of uncertain chaotic systems.

Feedback control based on a high-gain observer can induce undesirable dynamics effects such as the peaking phenomenon [22]. This phenomenon leads to closed-loop instabilities which are represented by finite-time escapes and large overshooting. To diminish these effects, the control law can be modified by means of

$$u = \mathrm{sat}\left\{ \int_0^t \left(\frac{1}{\hat{\gamma}(\hat{z})} \left[\beta S - \theta\,\mathrm{sgn}(S) - \hat{\eta} - \sum_{j=1}^{n+1} \theta^{(n-j+1)} K_j \hat{z}_j \right] \right) dt \right\}, \qquad (24)$$

where $\mathrm{sat}\{\cdot\}: \mathbb{R}^n \to S \subset \mathbb{R}^n$, S is a bounded set [6].

A similar synchronization scheme to that described above has been studied previously [19]. A drawback of such a scheme is that it is not apparent how one chooses the estimated value of the system's uncertainties so that the sliding surface and the control law become physically realizable. Our procedure has no such drawbacks. In fact, an estimate of the uncertainties is obtained via the new states z_{n+1} and η by means of a state estimator. Stability is guaranteed for sufficiently large values of the switching gain θ.

In the next section, we will show that the control strategy (20), (22), and (23) can be used to address problems of synchronization of chaos. In fact, we will illustrate via numerical simulations that the previously developed control strategy is able to synchronize uncertain chaotic systems with only knowledge of the output $\tilde{y}$.

IV. ILLUSTRATIVE EXAMPLES

We present two examples in this section to illustrate the above given results. The first example consists of the synchronization of two strictly different oscillators. The aim is to show that the synchronization can be attained in spite of model differences between the drive and response systems. We choose two second-order driven oscillators to illustrate this case. The drive system is given by the ϕ^6-Duffing equation, whereas the response system is given by the ϕ^6–Van der Pol oscillator. The second example consists of the synchronization of two Lorenz systems whose model is similar but whose parameter values are different. Here, the objective is to show that the synchronization can be achieved in spite of parameter variations and to illustrate that the chaotic minimum-phase assumption is satisfied.

A. Synchronization in spite of a strictly different model

The goal of this example is to illustrate that the synchronization can be attained in spite of a different model for the drive and response systems, which is the extreme case of drive/response mismatch and external perturbations by an oscillatory signal which can be interpreted as noise. We choose the ϕ^6-Duffing oscillator as the drive system and the ϕ^6–Van der Pol oscillator as the response system. The equations of the drive system are given as follows:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = f(x,t), \qquad (25)$$

where $f(x,t) = -r_1 x_2 - r_2 x_1 - r_3 x_1^3 - r_4 x_1^5 + r_5 \cos \omega t$. The equations of the response system are given by

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = g(y,t) + u, \qquad (26)$$

where $g(y,t) = \mu(1-y_1^2)y_2 - \omega_0^2 y_1 - \delta y_1^3 - \lambda y_1^5 + f_0 \cos \Omega t$ and u is the control input which has to be chosen. If $y_d = x_1$ and $y_r = y_1$ are, respectively, the outputs of the drive and response systems, and by defining $e_i = y_i - x_i$, $i = 1$, 2, one gets the following uncertain system:

$$\dot{e}_1 = e_2, \qquad \dot{e}_2 = g(e+x,t) - f(x,t) + u, \qquad y_e = e_1. \qquad (27)$$

Thus, the coordinates transformation is given by $z_1 = e_1$ and $z_2 = e_2$. In this way, system (27) is transformed into

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = \alpha(z,x,t) + u, \qquad \tilde{y} = z_1, \qquad (28)$$

where $\tilde{y}$ is the output of the uncertain system and $\alpha(z,x,t) = g(z+x,t) - f(x,t)$ denotes the uncertainties of these two systems (parameter mismatching and external perturbations). Note that system (27) is fully linearizable, i.e., there is no unobservable states ν in the uncertain system because the relative degree $\rho = n$. Now, defining $z_3 = \alpha(z,x,t) + u$ and $\eta = (\partial\alpha/\partial z_1)z_2 + (\partial\alpha/\partial z_2)z_3 + (\partial\alpha/\partial x_1)x_2 + (\partial\alpha/\partial x_2)f(x,t) + (\partial\alpha/\partial t)$, the dynamical system (6) can be constructed. So the extended state observer (20) can be described in the following form:

$$\begin{aligned}\dot{\hat{z}}_1 &= \hat{z}_2 - \theta C_1(\hat{z}_1 - z_1),\\ \dot{\hat{z}}_2 &= \hat{z}_3 - \theta^2 C_2(\hat{z}_1 - z_1),\\ \dot{\hat{z}}_3 &= \hat{\eta} - \theta^3 C_3(\hat{z}_1 - z_1) + \dot{u},\\ \dot{\hat{\eta}} &= -\,\theta^4 C_4(\hat{z}_1 - z_1).\end{aligned} \tag{29}$$

Hence the sliding surface and sliding control law (22) and (23) can be described by

$$S = \hat{z}_3 - \hat{z}_3(0) + \int_0^t [\theta^3 K_1 \hat{z}_1 + \theta^2 K_2 \hat{z}_2 + \theta K_3 \hat{z}_3]dt \tag{30}$$

and

$$u(t) = \int_0^t [\beta S - \theta\,\mathrm{sgn}(S) - \hat{\eta} - \theta^3 K_1 \hat{z}_1 - \theta^2 K_2 \hat{z}_2 - \theta K_3 \hat{z}_3]dt. \tag{31}$$

To ensure that both systems are chaotic, we select the parameter values $r_1=1$, $r_2=1$, $r_3=-3$, $r_4=1.5$, $r_5=480$, $\omega=1.221$, $\mu=0.4$, $\omega_0=0.46$, $\delta=1$, $\lambda=0.1$, $f_0=4.5$, and $\Omega=0.86$. Initial conditions for the ϕ^6-Duffing and ϕ^6–Van der Pol oscillators were selected as $x_1(0)=0$, $x_2(0)=0$, $y_1(0)=0.1$, and $y_2(0)=0$. Then $e_1(0)=0.1$ and $e_2(0)=0$. This choice of initial conditions is arbitrary: control can be applied for any initial conditions. The initial condition for $(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{\eta})$ is randomly chosen as (0.2, 0, 0, 0). The initial condition for the sliding surface (29) is $S(0)=0$ and $u(0)=0$ for the control input (31). The eigenvalues corresponding to the sliding surface are -10, $-10+6i$, and $-10-6i$, of which the coefficients of the Hurwitz polynomial are $[K_1,K_2,K_3]=[30,336,1360]$. The estimation parameters $C_1=4$, $C_2=6$, $C_3=4$, and $C_4=1$ were chosen so that the polynomial $s^4+C_1s^3+C_2s^2+C_3S+C_4=0$ has all its roots located at -1.

Let $\theta=50$. If the control input is activated at $t=15$ s, the synchronization error can be regulated effectively and efficiently to zero, as shown in Fig. 1. The performance of the error system is presented in Fig. 2. The corresponding control input is continuous, as shown in Fig. 2(a). The resulting control does not have an abrupt change and chattering phenomenon. The sliding surface dynamics is shown in Fig. 2(b).

To illustrate the fact that an arbitrary convergence rate of the synchronization error can be prescribed, Fig. 3 presents the position of the synchronization error $e_1=y_1-x_1$ for three different values of the switching gain θ. As expected, e_1 converges to zero and the larger the value of θ, the faster the convergence.

After the synchronization of the transmitter (drive) and receiver (response), one would like to know if a message signal can be recovered in spite of model differences between the transmitter and receiver. The information signal was chosen to be a periodic function $S_m(t)=1.5\sin 20t$. The frequency was chosen such that the dynamics behavior of the drive system remains chaotic. Figure 4 shows the time response of the error between $S_m(t)$ and the recovery signal $S_R(t)$ when $\theta=50$. The message signal is decoded with acceptable accuracy.

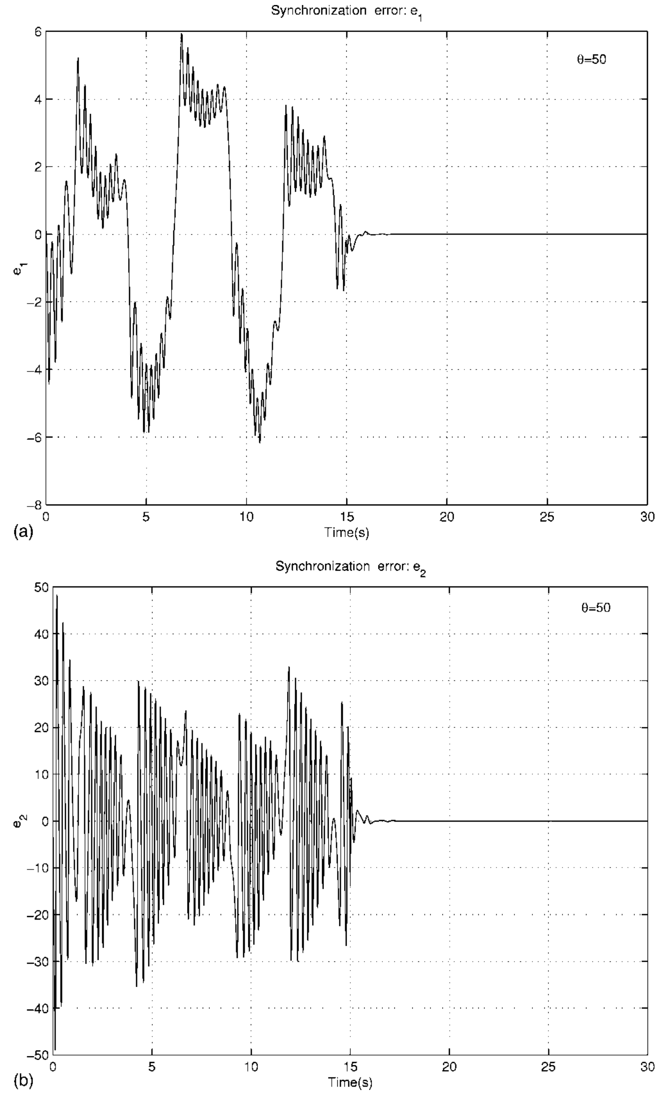

FIG. 1. Dynamics of the synchronization error when $\theta=50$. (a) Position e_1. (b) Velocity e_2.

B. Synchronization in spite of parametric variation

Here, the aim is to show that synchronization can be achieved in spite of parametric drive/response mismatches. The Lorenz system has been chosen to illustrate the proposed synchronization scheme. The Lorenz system consists of three simple nonlinearly coupled ordinary differential equations that depend on three positive parameters obtained from the Navier-Stokes equations for viscous fluids, originally derived for studying large-scale atmospheric behavior. It can be numerically integrated and the unexpected results have initiated the new and ubiquitous field of deterministic chaos, which occurs in many branches of physical, mathematical, biological, as well as social sciences.

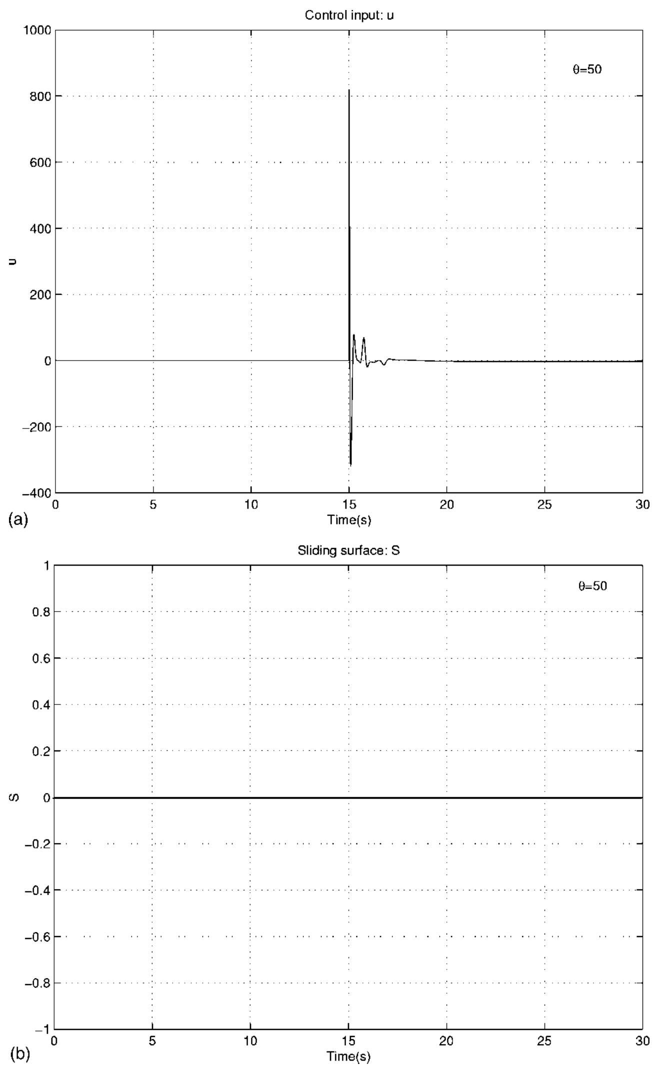

FIG. 2. Performance of the error system when $\theta=50$. (a) Control input. (b) Sliding surface dynamics.

The drive system can be written in dimensionless form as follows:

$$\dot{x}_1 = a_1(x_2 - x_1),$$

$$\dot{x}_2 = b_1x_1 - x_2 - x_1x_3, \tag{32}$$

$$\dot{x}_3 = x_1x_2 - c_1x_3.$$

Suppose that the same configuration is used as a response system. However, assume that there are differences between the devices. That is, the parameter values of the response system are different from the drive system. In this way, the response system becomes

$$\dot{y}_1 = a_2(y_2 - y_1) + u,$$

$$\dot{y}_2 = b_2y_1 - y_2 - y_1y_3, \tag{33}$$

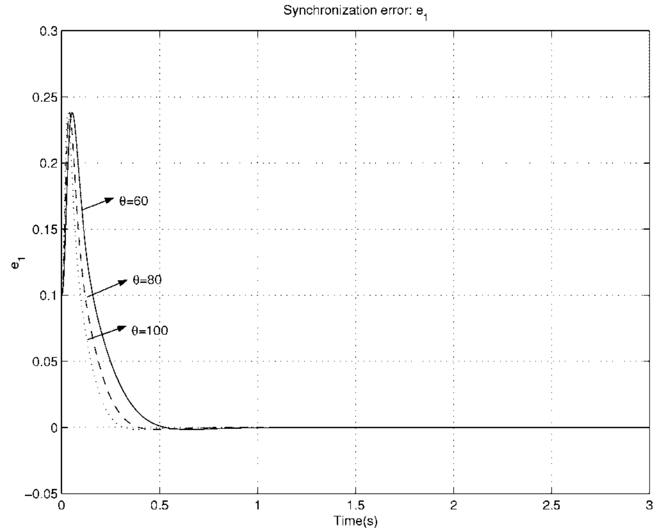

FIG. 3. Dynamics of the synchronization error $e_1=y_1-x_1$ for three different values of θ.

$$\dot{y}_3 = y_1y_2 - c_2y_3,$$

where u is the control input which has to be chosen. From the differences $e_i=x_i-y_i$, $i=1, 2, 3$, the uncertain system (3) can be obtained as follows:

$$\dot{e}_1 = a_1(e_2 - e_1) + (a_1 - a_2)(y_2 - y_1) - u,$$

$$\dot{e}_2 = b_1e_1 - e_2 - e_1(e_3 + y_3) + (b_1 - b_2 - e_3)y_1, \tag{34}$$

$$\dot{e}_3 = e_1(e_2 + y_2) + y_1e_2 - c_1e_3 + (c_2 - c_1)y_3.$$

Now defining the drive output by $y_d=x_1$ and the response output by $y_r=y_1$, one has that $y_e=e_1$. This implies that coordinates transformation is globally defined by $z_1=e_1$, $\nu_1=e_2$, and $\nu_2=e_3$. In this way, the smallest integer is $\rho=1$. Then the uncertain system can be rewritten as

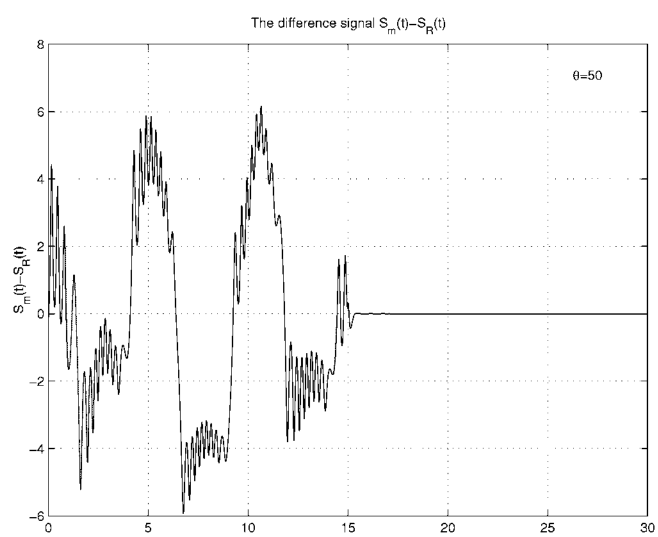

FIG. 4. The difference signal $S_m(t)-S_R(t)$ when $\theta=50$.

$$
\begin{aligned}
\dot{z}_1 &= \Delta g_1 - u,\\
\dot{\nu}_1 &= \Delta g_2,\\
\dot{\nu}_2 &= \Delta g_3,\\
\tilde{y} &= z_1,
\end{aligned} \tag{35}
$$

where $\tilde{y}$ is the output of the uncertain system and Δg_i, $i=1$, 2, 3 are unknown functions. In order to illustrate that system (35) satisfies the minimum phase assumption, one can show that $\Delta g_2 = b_1 z_1 - \nu_1 - z_1(\nu_2 + y_3) + \delta_1$ and $\Delta g_3 = z_1(\nu_1 + y_2) + y_1\nu_1 - c_1\nu_2 + \delta_2$, where $\delta_1 = (a_1 - a_2)(y_2 - y_1)$ and $\delta_2 = (c_2 - c_1)y_3$ converge to zero when $z_1 = 0$. Now, δ_1 and δ_2 are uncertain; however, it is clear that δ_1 and δ_2 are bounded. As $z_1 \to 0$ (zero dynamics), one has that

$$\dot{\nu} = E\nu + F,$$

where $F = [\delta_1, \delta_2]^T$ and

$$E = \begin{bmatrix} -1 & 0 \\ 0 & -c_1 \end{bmatrix},$$

which is Hurwitz because $c_1 > 0$. Hence, since δ_1 and δ_2 are bounded, the zero dynamics subsystem $\dot{\nu} = E\nu + F$ is asymptotically stable. That is, the discrepancy between systems (32) and (33) is a minimum phase system. Since assumptions (A_1)–(A_3) are satisfied, defining $z_2 = \Delta g_1 - u$, the augmented state can be defined as $\eta = (\partial\Delta g_1/\partial z_1)\Delta g_1 + (\partial\Delta g_1/\partial\nu_1)\Delta g_2 + (\partial\Delta g_1/\partial y_1)[a_2(y_2 - y_1) + u] + (\partial\Delta g_1/\partial y_2)(b_2 y_1 - y_2 - y_1 y_3)$. Then, system (6) can be constructed and we get the extended state observer (20) as the following form:

$$
\begin{aligned}
\dot{\hat{z}}_1 &= \hat{z}_2 - \theta C_1(\hat{z}_1 - z_1),\\
\dot{\hat{z}}_2 &= \hat{\eta} - \theta^2 C_2(\hat{z}_1 - z_1) - \dot{u},\\
\dot{\hat{\eta}} &= -\theta^3 C_3(\hat{z}_1 - z_1).
\end{aligned} \tag{36}
$$

So the sliding surface and sliding control law (22) and (23) can be described by

$$S = \hat{z}_2 - \hat{z}_2(0) + \int_0^t [\theta^2 K_1 \hat{z}_1 + \theta K_2 \hat{z}_2]dt \tag{37}$$

and

$$u(t) = -\int_0^t [\beta S - \theta\,\mathrm{sgn}(S) - \hat{\eta} - \theta^2 K_1 \hat{z}_1 - \theta K_2 \hat{z}_2]dt. \tag{38}$$

Here we choose the initial condition $S(0)=0$ and $u(0)=0$. The initial condition for the Lorenz system is (0.3, 0, 0). The initial condition for $(\hat{z}_1, \hat{z}_2, \hat{\eta})$ is randomly chosen as (0.1, 0, 0). The eigenvalues corresponding to the sliding surface are −1 and −2, of which the coefficients of the Hurwitz polynomial are $[K_1, K_2] = [2, 3]$. The estimation parameters $C_1 = 3$, $C_2 = 3$, and $C_3 = 1$ were chosen in such a way that the polynomial $s^3 + C_1 s^2 + C_2 s^1 + C_3 = 0$ has all its roots located at −1.

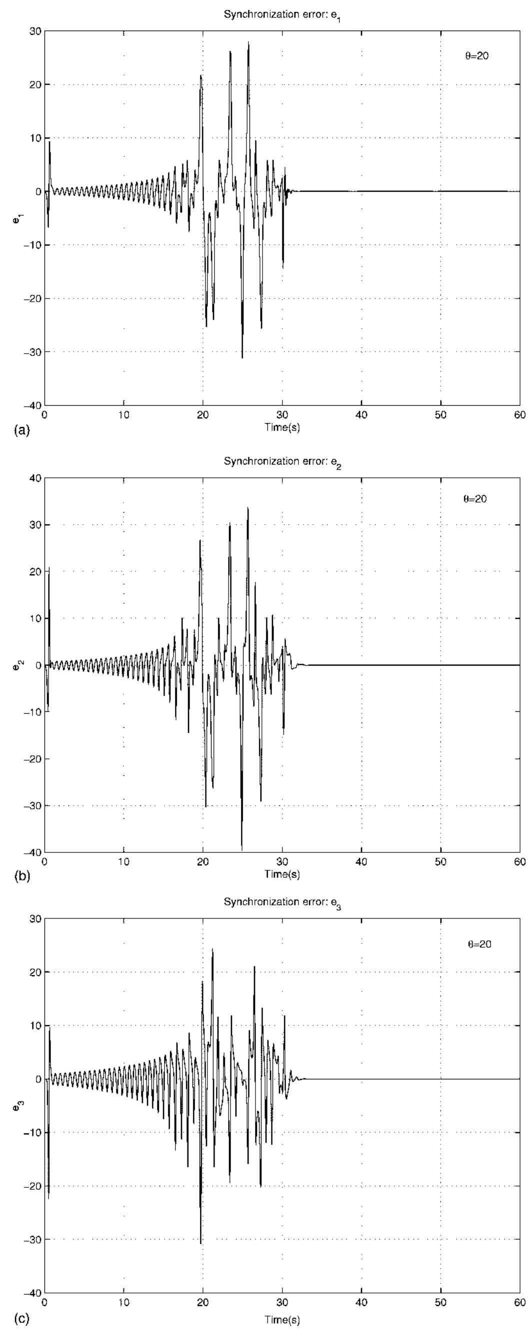

FIG. 5. Dynamics of the synchronization error of the Lorenz system when $\theta = 20$.

Simulation results are shown in Fig. 5. Although the control input u is acting on the state z_1, $\nu \in \mathbb{R}^2$ is also stabilized.

V. CONCLUSIONS

In this paper, the effort can be classified as follows. (i) A general mathematical model for chaotic systems is formulated, which contains most well-known continuous-time chaotic systems as special cases. (ii) The sliding mode control technique is combined with the chaos theory. (iii) A sliding surface was given in terms of a single parameter, which can be easily tuned to trade off between stability (convergence rate) and performance (noise, amplification). (iv) A new feedback controller is proposed for handling the uncertainties, both internal and external, existing in the chaotic dynamics. The control input in this study is continuous and has no chattering phenomenon. It provides a method that can achieve desired specification with less control energy by comparing against the results of other research. (v) An observer is used for estimating those unmeasured and model uncertainties but necessary information about the system state variables. We finally point out that to realize chaos synchronization via output feedback, various observers may be designed, which is beyond the scope of this paper.

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